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Least Squares Optimal Linearization

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Optimal linearization approximates a nonlinear system function by a best linear model over a specified region of states and controls. It is designed to improve upon conventional linearization and is different from other methods of linearization. This paper presents the theoretical solutions to square norm optimal linearization, derives a numerical algorithm for its implementation, and discusses its use in feedback design. The least squares optimal linearization has a unique solution over a hypercubic region. The optimal linear matrices are proper multiple integrals of the nonlinear function. Conventional linearization is the limiting case of optimal linearization where the region of interest is very small. In particular, the numerical algorithm computes conventional linear matrices efficiently. In general, optimal linear models are closer to given nonlinear systems than conventional linear models. Three examples are given to demonstrate the concept of optimal linearization. Optimal linearization applies to any continuous nonlinear functions and gives rise to sound numerical methods. Overall, it presents a new framework for linearizing nonlinear systems.

Introduction

ALL practical systems are nonlinear; however, linear system theories are much more mature than their nonlinear counterparts. In applying linear system theories to a nonlinear system, one needs to either approximate the nonlinear system by a linear system or transform it into a linear system. Optimal linearization determines a best linear approximation to the nonlinear function over a specified region of states and controls.

Conventional linearization keeps the linear term in a Taylor series expansion of a nonlinear function around a reference condition. The resulting linear model is descriptive of the nonlinear system only for sufficiently small variations of states and controls from the reference condition. In addition, the nonlinear function must be at least continuously differentiable at the reference point.[1]

In practical applications, linear feedback is applied over finite ranges of states and controls and to nonlinear functions that may not be continuously differentiable. We need to define another method of linearization, which works over definite ranges of states and controls and is applicable to continuous functions.

The answer is optimal linearization. Over specified ranges of states and controls, optimal linearization finds a best linear approximation to a given nonlinear function. Therefore, it produces a linear model that is closer to the nonlinear function than the conventional linear model. The least squares optimal linearization applies to any continuous functions and results in a numerical linearization method based on integration.

The proposed method of linearization differs from statistical linearization, the describing function method, and feedback linearization. Statistical linearization[2–4] requires the input signals to be Gaussian, whereas the describing function method[5–7] applies to sinusoidal input. There is no need to specify the input type in optimal linearization. Feedback linearization[7,8] transforms a nonlinear system into a linear one, whereas optimal linearization approximates a nonlinear system by a linear system.

Feedback controls of nonlinear systems may be achieved by gain scheduling. In this method, a nonlinear system is linearized around a series of operating conditions. A linear feedback control is designed for each of the operating conditions. Clearly, optimal linearization can be used instead of the conventional method to linearize the nonlinear system at each condition.

The idea of square norm optimal linearization has been previously used by two Yugoslavian scholars[9] in numerical linearization. However, the scope of our research is more comprehensive. On the theoretical side, this paper defines optimal linearization as an approximation problem, postulates that optimal linearization applies to any continuous functions, and presents the limiting case result. On the numerical side, by specializing the regions of linearization into hypercubes, this paper obtains a faster and more accurate numerical method. In addition, this paper advocates optimal linearization as an active step in the feedback design process.

Next we give a problem statement, a discussion of the least squares optimal linearization and special solutions over hybercubic regions, and a numerical linearization method. Example problems are given to demonstrate the use of optimal linearization, followed by discussions and conclusions.

Problem Statement

Consider a nonlinear system described by an ordinary differential equation.

$$\dot{X} = F(X, U) \tag{1}$$

where X is the $n \times 1$ state vector and U is the $m \times 1$ control vector. $F(X, U)$ is continuous in X and U. (X_e, U_e) is an equilibrium point; that is $F(X_e, U_e) = 0$. In Eq. (2), $x \triangleq X - X_e$, $u \triangleq U - U_e$, and the subscript $(\ )_e$ indicates evaluations at equilibrium.

Optimal linearization determines constant matrices $A \in R^{n\times n}$ and $B \in R^{n\times m}$ by minimizing

$$I = \|F(X_e + x, U_e + u) - Ax - Bu\| \tag{2}$$

for $x \in D_x$ and $u \in D_u$, where D_x and D_u are regions of interest for states and controls, and $\|\cdot\|$ is a certain norm for vector-valued functions over $\{D_x, D_u\}$. As a result, one obtains a linear model:

$$\dot{x} = Ax + Bu + g(x, u) \tag{3}$$

where the chosen functional norm of the error term $g(x, u)$ is minimized: $\|g(x, u)\|_{\min} \triangleq I_0$. I_0 depends on the function $F(X, U)$ and the region of interest.

The word "static" may be used because the proposed method approximates a nonlinear function itself by a linear one. Dynamic optimal linearization,[10] on the other hand, finds a linear system that best approximates the response of a nonlinear system under specified conditions.

The optimal linear model in Eq. (3) is actually a best linear approximation to the vector-valued function $F(X, U)$ of many variables. Approximation theories for scalar functions of one real variable

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are mature.[11,12] Multivariate approximation theory has experienced substantial development over the past decade.[13,14] Unfortunately, approximation theory for vector-valued functions of many variables has scarcely been studied.

Least Squares Optimal Linearization

The least squares optimal linearization finds constant matrices A and B, which minimize

$$I(A, B) = \|F(X_e + x, U_e + u) - Ax - Bu\|_2^2$$

$$\triangleq \int_{D_u} \int_{D_x} [F - Ax - Bu]^T [F - Ax - Bu] \, \mathrm{d}x \, \mathrm{d}u \tag{4}$$

where $(\,)^T$ represents the transpose of a matrix. The integrations are appropriate multiple integrations. Both D_x and D_u contain the zero vectors.

In the linearization process, the distinction between state and control is not really necessary. Therefore, we define

$$Z \triangleq \begin{bmatrix} X \\ U \end{bmatrix} \quad Z_e \triangleq \begin{bmatrix} X_e \\ U_e \end{bmatrix} \quad z \triangleq \begin{bmatrix} x \\ u \end{bmatrix} \in R^r \tag{5}$$

and

$$P = [A \quad B] \in R^{n \times r} \tag{6}$$

where $r = n + m$. The above problem is then equivalent to minimizing

$$I(P) = \|F(Z_e + z) - Pz\|_2^2$$

$$\triangleq \int_{D_z} [F(Z_e + z) - Pz]^T [F(Z_e + z) - Pz] \, \mathrm{d}z \tag{7}$$

where D_z is the region of interest for z containing $z = 0$.

To derive necessary conditions, we decompose $P \in R^{n \times r}$ into columns,

$$P = [p_1 \quad p_2 \quad \dots \quad p_r] \tag{8}$$

where $p_k \in R^n$, for $k = 1, \dots, r$. From Eq. (7),

$$I(P) = \int_{D_z} [F^T F - 2F^T P z + z^T P^T P z] \, \mathrm{d}z$$

$$= \int_{D_z} \left[F^T F - 2 \sum_{j=1}^{r} z_j F^T p_j + \left(\sum_{i=1}^{r} z_i p_i^T \right) \left(\sum_{j=1}^{r} z_j p_j \right) \right] \mathrm{d}z$$

where $z^T = [z_1 \; z_2 \; \dots \; z_r]$. Applying Leibnitz's rule[15] for integral differentiations, we have for $k = 1, \dots, r$,

$$\frac{\partial I}{\partial p_k} = 2 \int_{D_z} \left[-z_k F^T + \left(\sum_{i=1}^{r} z_i p_i^T \right) z_k \right] \mathrm{d}z$$

$$= 2 \int_{D_z} \left[z_k z^T P^T - z_k F^T \right] \mathrm{d}z$$

As a result, the necessary conditions are, for $k = 1, \dots, r$,

$$\int_{D_z} z_k z^T \, \mathrm{d}z \, P^T = \int_{D_z} z_k F^T \mathrm{d}z \tag{9}$$

Stacking all of these conditions together for $k = 1, \dots, r$ and transposing, we obtain

$$P \int_{D_z} z z^T \, \mathrm{d}z = \int_{D_z} F(Z_e + z) z^T \, \mathrm{d}z \tag{10}$$

where the integration of a matrix is performed on every element of the matrix.

The problem in Eq. (4) is quadratic in the parameter matrices A and B. Therefore, its solution, if it exists, is unique. The solution exists if the matrix

$$\int_{D_z} z z^T \, \mathrm{d}z \tag{11}$$

is not singular. Actually, the matrix is positive definite if it is not singular. This is true if the volume of the region D_z is not zero. The unique solution to the least squares optimal linearization is

$$P = \left(\int_{D_z} F(Z_e + z) z^T \, \mathrm{d}z \right) \left(\int_{D_z} z z^T \, dz \right)^{-1} \tag{12}$$

The use of the Leibnitz rule in this derivation requires that the nonlinear function be continuous. Actually, the linearization rule in Eq. (12) applies to any function that is Lebesgue measurable and square integrable. One can derive the same result without using the Leibnitz rule. The corresponding integrations are then defined in the Lebesgue sense. However, almost all numerical integration methods are based on Riemann integrals and require integrand functions to be piecewise continuous. Because most practical systems can be assumed to be continuous, we shall limit the use of optimal linearization to continuous functions.

Special Solutions

We often have some idea about the ranges of states and controls in which the linearized model is expected to operate. These ranges determine the structure of the integration region D_z.

Symmetric Regions

If the specified range for each state and control is symmetric with respect to the equilibrium point, we have a symmetric region:

$$D_S = \{z : -L_i \le z_i \le L_i \quad i = 1, \dots, r\} \tag{13}$$

where $L_i > 0$ is half the length of the interval for z_i.

On a symmetric region, Eq. (12) can be simplified. Define

$$L \triangleq \begin{bmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_r \end{bmatrix} \tag{14}$$

The unique solution to the problem in Eq. (4) over a symmetric region D_z is given by

$$P = \frac{3}{V} \int_{D_S} F(Z_e + z) \, z^T \, \mathrm{d}z \, L^{-2} \tag{15}$$

where V is the volume of the region D_S:

$$V = 2^r \prod_{j=1}^{r} L_j \tag{16}$$

Results of Eq. (15) also can be expressed in terms of A and B matrices. Define regions

$$D_{xS} = \{x\colon -L_{x_i} \le x_i \le L_{x_i} \quad i = 1, \dots, n\} \tag{17}$$

$$D_{uS} = \{u\colon -L_{u_j} \le u_j \le L_{u_j} \quad j = 1, \dots, m\} \tag{18}$$

and

$$L_x \triangleq \begin{bmatrix} L_{x_1} & 0 & \cdots & 0 \\ 0 & L_{x_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_{x_n} \end{bmatrix} \tag{19}$$

$$L_u \triangleq \begin{bmatrix} L_{u_1} & 0 & \cdots & 0 \\ 0 & L_{u_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_{u_m} \end{bmatrix} \tag{20}$$

we have

$$A = \frac{3}{V} \int_{D_{uS}} \int_{D_{xS}} F(X_e + x, U_e + u) x^T \, dx \, du \, L_x^{-2} \tag{21}$$

$$B = \frac{3}{V} \int_{D_{uS}} \int_{D_{xS}} F(X_e + x, U_e + u) u^T \, dx \, du \, L_u^{-2} \tag{22}$$

where

$$V = 2^{n+m} \prod_{i=1}^{n} L_{x_i} \prod_{j=1}^{m} L_{u_j} \tag{23}$$

General Hypercubes

Sometimes the range of interest for a state or a control may not be symmetric with respect to the reference point. This can happen if there are bounds on the state and/or control variables. In general, we have a hypercubic region of integration:

$$D_H = \{z: -c_i \le z_i \le d_i, \quad i = 1, \ldots, r\} \tag{24}$$

where $c_i \ge 0$, $d_i \ge 0$, and $c_i^2 + d_i^2 \ne 0$.

To obtain the solution in this case, we define, for $i = 1, \ldots, r$,

$$C_i = \tfrac{1}{2}(d_i - c_i) \quad L_i = \tfrac{1}{2}(d_i + c_i) \tag{25}$$

In this definition, C_i is the center coordinate and L_i is the half length of the interval $[-c_i, d_i]$. We also define

$$\overline{C}_i \triangleq \frac{C_i/L_i}{\sqrt{1/3 + \sum_{j=1}^{r} (C_j/L_j)^2}} \quad i = 1, \ldots, r \tag{26}$$

and

$$\overline{C} = \begin{bmatrix} \overline{C}_1 \\ \overline{C}_2 \\ \vdots \\ \overline{C}_r \end{bmatrix} \tag{27}$$

After some manipulations, we find

$$P = \frac{3}{V} \int_{D_H} F(Z_e + z) z^T dz L^{-1} (I - \overline{C}\,\overline{C}^T) L^{-1} \tag{28}$$

where I is the $r \times r$ identity matrix. For a symmetric region, $\overline{C} = 0$ and this solution reduces to Eq. (15). Just as in the case of symmetric regions, one can express Eq. (28) in terms of A and B.

A given nonlinear system may be linear in some variables. For example, kinematic equations may be linear in states. Also, certain state and/or control variables may appear linearly in some but not all of the system equations. In all these cases, optimal linearization amounts to finding the averages of the linear variable coefficients. In particular, constant coefficients of linear variables remain the same in the least squares optimal linearization process. Details are discussed in Zhao.[16]

Limiting Case Result

Conventional linearization is actually a special case of optimal linearization. As the region of interest becomes very small around the reference point, results of optimal linearization approach those of conventional linearization. In particular, we have the following limiting case result.

If the region of interest is symmetric, and f is continuously differentiable at $z = 0$,

$$\lim_{|L| \to 0} P = \frac{\partial F}{\partial Z}(Z_e) \qquad \text{where } |L| \triangleq \max_{1 \le i \le r} L_i \tag{29}$$

A proof is sketched in Ref. 16. A similar result holds for general regions. One can prove this by enclosing a given region with two symmetric regions. As a result, one can use optimal linearization to obtain conventional linear models.

Numerical Methods

The calculation of optimal linear matrices involves multiple integrations. Evaluations of multiple integrals heavily depend on the associated regions. Fortunately, we can transform our multiple integrations over hypercubes onto the simplest and most-studied region: a unit hypercube.

One-dimensional numerical integration is a mature subject. For a given function $h(x)$ over $[a, b]$, a rule of numerical integration refers to

$$\int_a^b h(x) \, dx \approx \sum_{i=1}^{N} w_i h(\xi_i) \tag{30}$$

where ξ_i represents the abscissas and w_i the weights of the integration rule. There are different families of integration rules (Davis and Rabinowitz[17]). If both the abscissas and weights are free to vary, Gauss type of rules are exact for the highest order $(2N - 1)$ polynomials for a fixed number (N) of abscissas. In these rules, the abscissas and weights are determined from zeros of a certain orthogonal polynomial; the simplest being the Legendre polynomial. Press et al. give a simple program to compute the abscissas and weights of Gauss-Legendre one-dimensional integration.[18]

There are three ways of performing multiple integrations.[17,19] A product rule decomposes a multiple integral into a series of one dimensional integrations. If an N-point one dimensional formula is used in a product rule for an $(n + m)$-dimensional multiple integration, there is a total of N^{n+m} function evaluations. Therefore, a product rule demonstrates a strong "dimensional effect." Rules exact for monomials are devised to integrate monomials exactly over a multiple-dimensional space. These rules require fewer function evaluations than the product rule. Multiple integrations of very high dimensions may be evaluated by sampling. The idea is to treat a multiple integral as a certain stochastic average of the integrand function. Over the last decade, there have been many developments in the calculations of multiple integrals.[20] Together with the advances in the speed of computation, these developments have made feasible the calculations of high-dimensional multiple integrations.

In this paper, we transform the solutions in Eqs. (21) and (22) to the unit hypercube, use a product rule, and employ the Gauss-Legendre one-dimensional formula. These choices work well for nonlinear functions of moderate dimension.

Define

$$\tilde{x} = L_x^{-1} x \qquad \text{and} \qquad \tilde{u} = L_u^{-1} u \tag{31}$$

we have from Eqs. (21) and (22),

$$A = \frac{3}{2^{n+m}} \int \int F(X_e + L_x \tilde{x}, U_e + L_u \tilde{u}) \tilde{x}^T \, d\tilde{x} \, d\tilde{u} L_x^{-1} \tag{32}$$

$$B = \frac{3}{2^{n+m}} \int \int F(X_e + L_x \tilde{x}, U_e + L_u \tilde{u}) \, \tilde{u}^T \, d\tilde{x} \, d\tilde{u} \, L_u^{-1} \tag{33}$$

where the two multiple integrations are performed over

$$\tilde{x} \in [-1, 1]^n \quad \tilde{u} \in [-1, 1]^m \tag{34}$$

Decompose A and B into columns

$$A = [a_1, a_2, \ldots, a_n] \tag{35}$$

$$B = [b_1, b_2, \ldots, b_m] \tag{36}$$

where a_i and b_j are $n \times 1$ column vectors, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. An optimal linearization algorithm based on product rule is then given by

$$\mathbf{a}_i \approx \frac{3}{2^{n+m} L_{x_i}} \sum_{k_1=1}^{N} \cdots \sum_{k_{n+m}=1}^{N} w_{k_1} \cdots w_{k_{n+m}} F(X, U) \xi_{k_i} \tag{37}$$

$$\mathbf{b}_j \approx \frac{3}{2^{n+m}L_{u_j}} \sum_{k_1=1}^{N} \cdots \sum_{k_{n+m}=1}^{N} w_{k_1} \cdots w_{k_{n+m}} F(X,U)\xi_{k_{n+j}} \quad (38)$$

where

$$\mathbf{X} = X_e + L_x \begin{bmatrix} \xi_{k_1} \\ \vdots \\ \xi_{k_n} \end{bmatrix} \quad (39)$$

$$\mathbf{U} = U_e + L_u \begin{bmatrix} \xi_{k_{n+1}} \\ \vdots \\ \xi_{k_{n+m}} \end{bmatrix} \quad (40)$$

A computer program based on these choices was coded in C language, where the abscissas ξ and the weights w are the zeros and values of an Nth-order Legendre polynomial. The program is about three pages long and produces a linear model for systems with $n+m \leq 15$ in a matter of seconds on a Sun workstation. Since the amount of computation is proportional to N^{n+m}, one should start with a small N for systems of higher orders.

Examples

We present three examples in the following. The first two examples provide analytical solutions that readers can derive by hand. In the third example, nonlinear equations of a rigid-body aircraft model are linearized.

First Order Example

Consider

$$\dot{x} = x^{\frac{1}{3}} + e^{-x}\sin u \quad (41)$$

which has an equilibrium at $x = 0$ and $u = 0$. The first term does not give a finite derivative at $x = 0$. Thus, a conventional linear model does not exist.

Optimal linearization leads to

$$\dot{x} = ax + bu \quad (42)$$

Over the region of $-c \leq x \leq c$ and $-d \leq u \leq d$, where $c > 0$ and $d > 0$, optimal linear model parameters a and b are determined from

$$\min_{a,b} \int_{-c}^{c}\int_{-d}^{d} (x^{\frac{1}{3}} + e^{-x}\sin u - ax - bu)^2 \, du \, dx \quad (43)$$

We have

$$a = \frac{9}{7c^{2/3}} \quad (44)$$

$$b = 1.5\,\frac{\sin d - d\cos d}{d^3}\,\frac{e^c - e^{-c}}{c} \quad (45)$$

In particular, as both c and d approach zero,

$$a \to \infty \quad b \to 1 \quad (46)$$

which agrees with conventional linearization.

A Second Order Example

Consider a second-order system

$$\dot{x}_1 = x_2 \quad (47)$$

$$\dot{x}_2 = -x_2\cos x_1 + x_1^3 + u \quad (48)$$

This system has an equilibrium at $x_1 = 0$, $x_2 = 0$, and $u = 0$.

Conventional linearization gives

$$\dot{x}_1 = x_2 \quad (49)$$

$$\dot{x}_2 = -x_2 + u \quad (50)$$

Because this system is already linear in x_2 and u, optimal linearization gives

$$\dot{x}_1 = x_2 \quad (51)$$

$$\dot{x}_2 = ax_1 - bx_2 + u \quad (52)$$

where a and b are determined from

$$\min_a I_1 = \int_{-c}^{c} (x_1^3 - ax_1)^2 \, dx_1$$

$$\min_b I_2 = \int_{-c}^{c} (\cos x_1 - b)^2 \, dx_1$$

which results in

$$a = 0.6c^2 \quad b = \sin c/c \quad (53)$$

where the region of interest is $-c \leq x_1 \leq c$. As $c \to 0$, $a \to 0$ and $b \to 1$. In other words, optimal linearization produces the same result as conventional linearization when the specified range goes to zero.

Over a finite region, the optimal linear model in Eqs. (51) and (52) is on average closer to the nonlinear system than the conventional linear model in Eqs. (49) and (50). To demonstrate this fact, let us design a feedback control law to stabilize the system at $x_1 = 0$ and $x_2 = 0$ for $x_1(0) \neq 0$. For this simple system, we can specify the closed-loop characteristics as follows:

$$\ddot{x}_1 + K_1\dot{x}_1 + K_2 x_1 = 0 \quad (54)$$

where $K_1 > 0$ and $K_2 > 0$ are feedback gain constants.

Using the conventional linear model in Eqs. (49) and (50), we obtain the following linear feedback control law

$$u_c = (1 - K_1)x_2 - K_2 x_1 \quad (55)$$

Using the optimal linear model in Eqs. (51) and (52), we get

$$u_o = (\sin c/c - K_1)x_2 - (0.6c^2 + K_2)x_1 \quad (56)$$

We now substitute the control laws into the original nonlinear system in Eqs. (47) and (48) one at a time. We assume $c = 0.8$, $K_1 = 2$, and $K_2 = 1$. Figure 1 plots the responses of the nonlinear system for $x_1(0) = 0.2, 0.8, 1.01$. In this figure, "optimal" refers to responses using the feedback based on the optimal linear model in Eq. (56), "conventional" refers to responses using Eq. (55), and "linear" refers to responses of Eq. (54). Indeed, the linear feedback control law designed from the optimal linear model is on average closer to linear predictions.

A Rigid-Body Aircraft Model

The following equations describe a rigid-body aircraft in two-dimensional vertical motion.

$$m\dot{V} = T\cos\alpha - D - mg\sin\gamma \quad (57)$$

$$mV\dot{\gamma} = T\sin\alpha + L - mg\cos\gamma \quad (58)$$

$$\dot{\alpha} = q - \dot{\gamma} \quad (59)$$

$$I_{yy}\dot{q} = M - x_c L\cos\alpha - x_c D\sin\alpha \quad (60)$$

In these equations, m is the mass, T the thrust, α the angle of attack, γ the flight path angle, (L, D) are the lift and drag force, q is the pitch rate, I_{yy} is the moment of inertia about the y-axis, M is the pitching moment, and x_c is the distance between the aircraft aerodynamic center and the center of mass. If the aerodynamic center is behind the center of mass, $x_c > 0$.

Lift, drag, and pitching moment are defined through corresponding coefficients: $L = 0.5\rho V^2 S C_L$, $D = 0.5\rho V^2 S C_D$, and $M = 0.5\rho V^2 S\bar{c}C_M$, where ρ is the air density, S is the reference area, and $\bar{c}$ is the mean aerodynamic chord. The aerodynamic data given

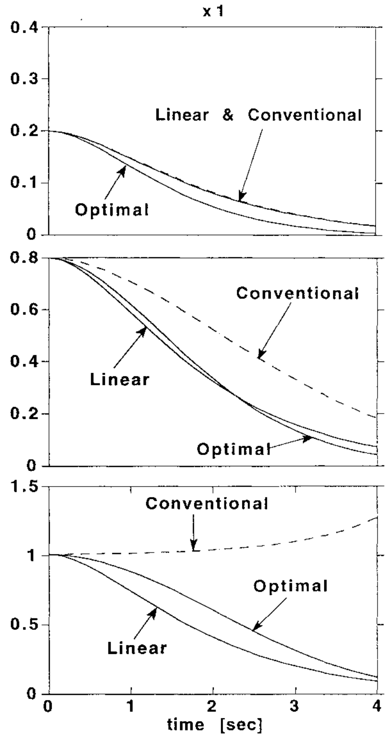

Fig. 1 Second-order example.

in the Appendix correspond to a hypothetical high-performance aircraft. In particular, the pitching motion is controlled through canard.

The aerodynamic coefficients are given by

$$C_L = C_L(\alpha) + (1/2.235)(\delta_c + \alpha)C_{\delta_c}(\alpha) \tag{61}$$

$$C_D = C_D(\alpha) \tag{62}$$

$$C_M = (\delta_c + \alpha)C_{\delta_c}(\alpha) - 1.5q\bar{c}/V \tag{63}$$

where δ_c is the canard deflection.

To achieve good numerical properties, we normalize the speed by 100:

$$v = V/100 \tag{64}$$

The normalized equations become

$$\dot{v} = \overline{T}\cos\alpha - Ev^2C_D - \overline{g}\sin\gamma \tag{65}$$

$$v\dot{\gamma} = \overline{T}\sin\alpha + Ev^2C_L - \overline{g}\cos\gamma \tag{66}$$

$$\dot{\alpha} = q - (\overline{T}\sin\alpha + Ev^2C_L - \overline{g}\cos\gamma)/v \tag{67}$$

$$\dot{q} = Fv^2(C_M - \overline{x}_c\cos\alpha C_L - \overline{x}_c\sin\alpha C_D) \tag{68}$$

where C_L and C_D are the same as in Eqs. (61) and (62), and C_M is given by

$$C_M = (\delta_c + \alpha)C_{\delta_c}(\alpha) - 0.066q/v \tag{69}$$

In these equations,

$$\overline{T} = T/100m \quad E = 50\rho S/m \quad \overline{g} = g/100 \quad F = 10^4\rho S\bar{c}/2I_{yy}$$

We linearize these equations around an equilibrium condition. The linear system is in the form of

$$\begin{bmatrix}\dot{v}\\ \dot{\gamma}\\ \dot{\alpha}\\ \dot{q}\end{bmatrix} = A\begin{bmatrix}v - v_0\\ \gamma - \gamma_0\\ \alpha - \alpha_0\\ q\end{bmatrix} + B\begin{bmatrix}\delta_c - \delta_{co}\\ \overline{T} - \overline{T}_0\end{bmatrix} \tag{70}$$

A program is written to determine equilibrium conditions. A set of equilibrium conditions are: $v_0 = 1$, $\gamma_0 = 0$, $\alpha_0 = 4.3143$ deg, $\delta_{co} = -7.0933$ deg, and $\overline{T}_0 = 0.012031$. The conventional linearization gives:

$$A = \begin{bmatrix}-0.0240 & -0.0980 & -0.1040 & 0\\ 0.1942 & 0 & 1.3815 & 0\\ -0.1942 & 0 & -1.3815 & 1\\ 0 & 0 & 9.6220 & -1.3312\end{bmatrix} \tag{71}$$

$$B = \begin{bmatrix}0 & 0.9972\\ 0.0419 & 0.0752\\ -0.0419 & -0.0752\\ 5.7946 & 0\end{bmatrix} = [b_1 \quad b_2] \tag{72}$$

The optimal linearization program produces a linear model very quickly with $N = 5$. With $L_v = 0.001$, $L_\gamma = 0.1$ deg, $L_\alpha = 0.1$ deg, $L_q = 0.1$ deg/s, $L_{\delta_c} = 0.1$ deg, and $L_t = 0.001$, the optimal linearization gives the same system matrices. Open loop eigenvalues are located at

$$-4.4600 \quad 1.7554 \quad -0.0161 \pm 0.1521j \tag{73}$$

If the ranges of states and controls are not very small, optimal linearization produces different results. With $L_v = 0.02$, $L_\gamma = 10$ deg, $L_\alpha = 30$ deg, $L_q = 10$ deg, $L_{\delta_c} = 40$ deg, and $L_t = 0.001$, the optimal linearization gives

$$A_o = \begin{bmatrix}-0.1482 & -0.0977 & -0.1045 & 0\\ 0.1698 & 0 & 1.1544 & 0\\ -0.1698 & 0 & -1.1544 & 1\\ -0.1326 & 0 & 8.8590 & -1.3312\end{bmatrix} \tag{74}$$

$$B_o = \begin{bmatrix}0 & 0.9522\\ 0.0413 & 0.0718\\ -0.0413 & -0.0718\\ 5.7124 & 0\end{bmatrix} = [b_{o1} \quad b_{o2}] \tag{75}$$

Since the nonlinear equations are linear in q, δ_c, and $\overline{T}$, the choices of L_q, L_{δ_c}, and L_t are not important. The corresponding open-loop eigenvalues are

$$-4.2216 \quad 1.7445 \quad -0.0783 \pm 0.1259j \tag{76}$$

which reflect a similar open loop characteristics.

Over this region, the optimal linear model is closer to the nonlinear aircraft equations than the conventional linear model. However, the conventional linear model and the optimal linear model represent two different linear systems. Even with eigenvalue assignments on the same eigenvalues, these two models produce two different closed-loop systems, or different linear predictions. Therefore, it is difficult to compare them directly. In the following, we design linear feedback control laws from each of the two models and substitute the control laws one at a time into the nonlinear equations. Each control law is then compared with its corresponding linear predictions. We are going to see that the control law based on the optimal linear model produces responses that are closer to its corresponding linear predictions.

Let us design a feedback control law to stabilize the short period mode. We can analyze the open-loop linear system properties using a modal analysis method.[21] The short period mode is unstable. It is mainly observable from α and to a lesser extent from q, and is mainly controllable from the canard deflection. The phugoid mode is stable though not well damped. It is observable from u and γ, and controllable from both thrust and canard. For simplicity, we set $\overline{T} = \overline{T}_0$ and use canard as the only control.

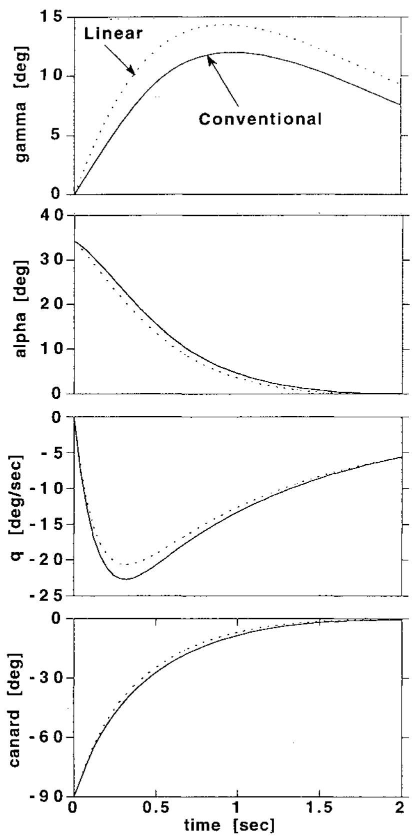

Fig. 2 Conventional linearization with $\Delta\alpha = 30$ deg.

The following LQR designs are performed with $Q = \text{diag}\,[1, 1, 1, 1]$ and $R = 1$. Feedback control laws are given as

$$\delta_c - \delta_{co} = -K \begin{bmatrix} v - v_0 \\ \gamma - \gamma_0 \\ \alpha - \alpha_0 \\ q \end{bmatrix} \tag{77}$$

Using the conventional linear model in Eqs. (71) and (72), we obtain

$$K = [-0.7737 \quad 1.0761 \quad 2.8488 \quad 1.2000] \tag{78}$$

The corresponding linear predictions are given by the closed-loop matrix $A_{\text{cl}} = A - b_1 K$. Using the optimal linear model in Eqs. (74) and (75), we have

$$K_o = [-0.2079 \quad 0.9932 \quad 2.7124 \quad 1.1854] \tag{79}$$

The corresponding linear predictions are given by $A_{o\text{cl}} = A_o - b_{o1} K_o$.

We substitute the two linear feedback control laws of Eqs. (78) and (79) into the nonlinear equations Eqs. (65–68) one at a time. Numerical simulations are performed for different values of $\Delta\alpha$ with the initial condition of $v(t_0) = 1.0$, $\gamma(t_0) = 0$, $\alpha(t_0) = \alpha_0 + \Delta\alpha$, and $q(t_0) = 0$. Two sets of simulations are presented. Figures 2 and 3 correspond to $\Delta\alpha = 30$ deg, whereas Figs. 4 and 5 correspond to $\Delta\alpha = 5$ deg.

The feedback design using the optimal linear model is on average closer to linear design predictions than that based on the conventional linear model. This is clear from the case of $\Delta\alpha = 30$ deg. When $\Delta\alpha = 5$ deg, the response with conventional linear control law is almost identical with its corresponding linear predictions, as expected. The optimal linear model, on the other hand, obtains an overall better description of the nonlinear system at the expense of nonexact matching at small angles of attack. With $\Delta\alpha = 15$ deg, the two control laws are on about the same level of closeness to their linear predictions. With $\Delta\alpha = 40$ deg, the optimal linear model is closer.

We also performed optimal linearizations for larger ranges of v, γ, and α, e.g., $L_\alpha = 80$ deg. When the ranges of interest are too large, the least squares optimal linear model does not describe the nonlinear system very well for small variations of state variables from equilibrium.

Discussion

Optimal linearization can be used efficiently to obtain the conventional linear model. Numerical integrations are in general more stable than differentiations. One can use small values of the half-lengths (L) to start an optimal linearization and then repeat the process with smaller half-lengths. If there is not much change in the linear matrices, the conventional linear model is found. This is guaranteed by the limiting case result. Usually, half-lengths on the order of 1% of expected variable ranges are good enough.

Optimal linearization achieves an overall good description of a given nonlinear system over a specified region. Conventional linearization, on the other hand, closely predicts a nonlinear system behavior only for small variations of states and controls from equilibrium. The usefulness of optimal linear models depends on the nonlinear function and the region of choice. If the region is not too large, optimal linearization always improves upon conventional linearization method.

In stabilizing a nonlinear system around an equilibrium condition, we can determine the ranges of states and controls caused by disturbances or initial conditions. Then, we can perform an optimal linearization over these ranges. The resulting optimal linear model can be used for linear feedback design. This process is repeated if necessary.

An important issue in the proposed linearization method is being studied. It is highly desirable if the stability of the linearized system can be related to that of the nonlinear system. We find that a minimax optimal linearization promises to offer a desirable stability guarantee. The least squares optimal linearization, on the other hand, does not easily lead to a stability relation. Still, the least squares optimal linearization presents a simple method for application. Its systematic uses in feedback design may be pursued in the following directions: 1) Optimal linear models over different regions exhibit variations

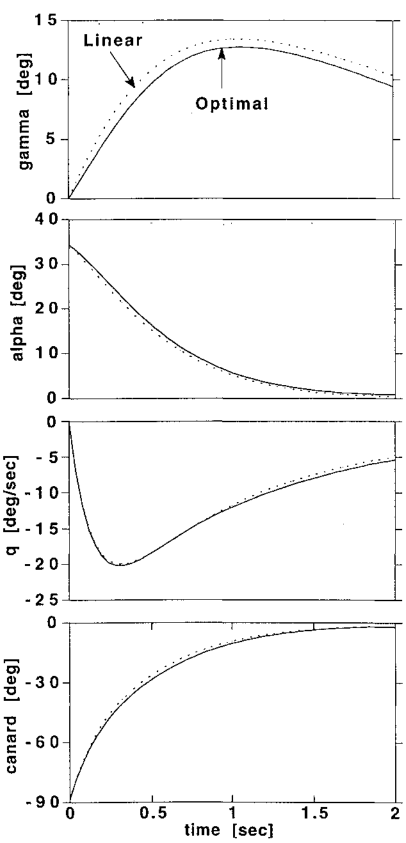

Fig. 3 Optimal linearization with $\Delta\alpha = 30$ deg.

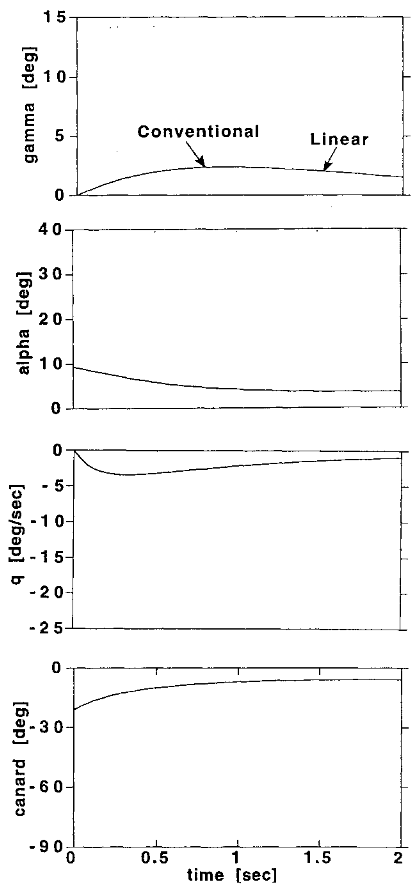

Fig. 4 Conventional linearization with $\Delta\alpha = 5$ deg.

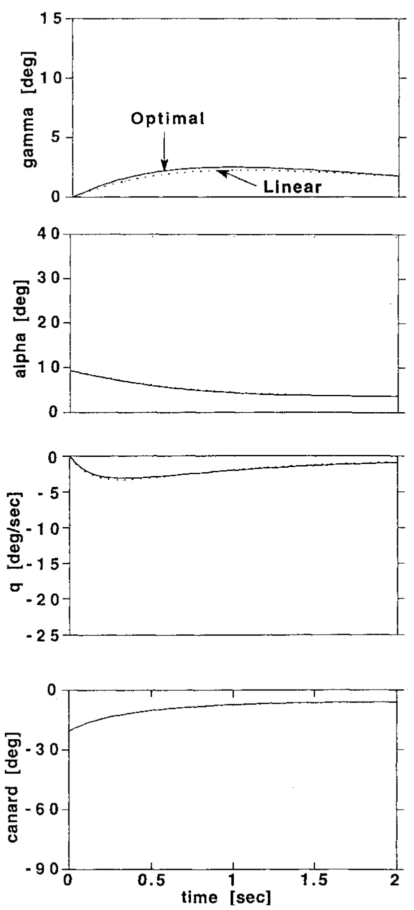

Fig. 5 Optimal linearization with $\Delta\alpha = 5$ deg.

in system parameters. In the rigid-body aircraft systems for example, $A(1, 1)$ changes over different regions. These variations can be treated as parameter uncertainties. There is a rich literature on linear robust control in uncertain parametric systems. 2) The process of optimal linearization produces a linear model and a bound on the linearization error: I_0. One can use linear robust control theories to stabilize the linear system in Eq. (3), subject to all unstructured *uncertainties* $g(x, u)$ bounded by $\|g(x, u)\| \leq I_0$. (3) Alternatively, one can treat $g(x, u)$ as an *intelligent adversary* with a bound I_0 and design a stabilizing controller via differential games.

Conclusion

This paper presents the theoretical solutions and a numerical algorithm for the least squares optimal linearization. Optimal linearization finds a best linear approximation to a nonlinear function over a specified region. There is a unique solution to the square norm optimal linearization over a hypercubic region. The optimal linear matrices are appropriate multiple definite integrals of the given nonlinear function. Conventional linearization is a special case of the least squares optimal linearization.

A numerical algorithm for performing least squares optimal linearization is provided. This algorithm transforms the optimal linearization onto the multiple dimensional unit cube and employs a product rule that repeatedly uses the one-dimensional Gauss-Legendre integration formula. A computer program implementing this algorithm is short and works well for nonlinear systems of moderate dimension. Specifically, one can quickly obtain the conventional linear model using this program.

Over a specified region, the least squares optimal linear model is on average closer to a given nonlinear system than the conventional linear model. As a result, linear feedback controls using optimal linear models, when applied to the nonlinear systems, are closer to linear design predictions than those of conventional linear models.

Appendix: Aircraft Data

The following data are taken from Snell[22] and fitted into fifth-order polynomials for $0 \leq \alpha \leq 80$ deg. Snell gathered these data from various sources for a hypothetical highly maneuverable aircraft. In particular, the lift and drag coefficients are given by Lallman,[23] and the canard function is provided by Fellers et al.[24]

For $\alpha > 0$,

$$C_L(\alpha) = A_0 + A_1\alpha + A_2\alpha^2 + A_3\alpha^3 + A_4\alpha^4 + A_5\alpha^5$$

$$C_D(\alpha) = B_0 + B_1\alpha + B_2\alpha^2 + B_3\alpha^3 + B_4\alpha^4 + B_5\alpha^5$$

$$C_{\delta_c}(\alpha) = 1.8/\pi(C_0 + C_1\alpha + C_2\alpha^2 + C_3\alpha^3 + C_4\alpha^4 + C_5\alpha^5)$$

where $A_0 = 0.00933$, $A_1 = 3.58977$, $A_2 = 4.40752$, $A_3 = -16.98693$, $A_4 = 13.38188$, $A_5 = -3.34885$; $B_0 = 0.02323$, $B_1 = 0.03809$, $B_2 = 1.64156$, $B_3 = 1.65442$, $B_4 = -2.30301$, $B_5 = 0.55977$; $C_0 = 0.50499$, $C_1 = -0.26789$, $C_2 = 1.31669$, $C_3 = -2.63005$, $C_4 = 1.87604$, $C_5 = -0.44978$. These expressions can be extended for $\alpha < 0$ as follows:

$$C_L(\alpha) = -C_L(|\alpha|) + 2A_0$$

$$C_D(\alpha) = C_D(|\alpha|)$$

$$C_{\delta_c}(\alpha) = C_{\delta_c}(|\alpha|)$$

Other parameters include: $m = 10{,}617$ kg, $I_{yy} = 77{,}095$ kg m^2, $S = 57.7$ m^2, $\bar{c} = 4.4$ m, $x_c = -0.0465\bar{c}$, and $\rho = 1.225$ kg/m^3. The canard deflection is bound by -90 deg $\leq \delta_c \leq 30$ deg. α is measured in radians.

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